THE COMPLETIONS OF METRIC ANR'S AND UNIFORM ANR'S (Research in General and Geometric)

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THE COMPLETIONS OF METRIC ANR'S AND UNIFORM ANR'S

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A subset Y of a space X is said to be *homotopy dense* in X if there exists a homotopy $h\colon X \times [0,1] \to X$ such that $h_0 = \mathrm{id}$ and $h_t(X) \subset Y$ for $t > 0$. This concept is very important in ANR Theory and Infinite-Dimensional Topology. When X is an ANR, the concept of the homotopy denseness is dual to the one of local homotopy negligibility introduced by Toruńczyk in [To$_3$], that is, $Y \subset X$ is homotopy dense in X if and only if the complement $X \setminus Y$ is locally homotopy negligible in X (cf. [To$_3$, Theorem 2.4]). The following fact is well-known:

Fact. *Every homotopy dense subset of an ANR is also an ANR and a metrizable space is an ANR if it contains an ANR as a homotopy dense subset.*

The lack of the homotopy denseness of a metric ANR in its completion often destroys the ANR property of the completion. For instance, the $\sin\frac{1}{x}$-curve in the plane $\mathbb{R}^2$ is an ANR but the completion of this curve ($=$ the closure in $\mathbb{R}^2$) is not an ANR. Moreover, even if the completion is an ANR, it is very different from the original ANR. The circle $\mathbf{S}^1$ is the completion of the space $\mathbf{S}^1 \setminus \{\mathrm{pt}\}$ and the both spaces are ANR but they are topologically very different from each other. It should be remarked that $\mathbf{S}^1 \setminus \{\mathrm{pt}\}$ is not homotopy dense in $\mathbf{S}^1$. In this note,[1] we consider the following interesting problem:

Problem. *When is a metric ANR homotopy dense in the metric completion?*

1. A CHARACTERIZATION OF METRIC ANR'S

The nerve of an open cover $\mathcal{V}$ of a space X is denoted by $N(\mathcal{V})$. A sequence $\mathcal{U} = (\mathcal{U}_n)_{n\in\mathbb{N}}$ of open covers of a metric space X is called a *zero-sequence* if $\lim_{n\to\infty} \operatorname{mesh} \mathcal{U}_n = 0$. For such a sequence, we define the simplicial complex

$$TN(\mathcal{U}) = \bigcup_{n\in\mathbb{N}} N(\mathcal{U}_n \cup \mathcal{U}_{n+1}),$$

[1] The results mentioned in this note were obtained in [Sa]. Then, for details, one can refer to the paper [Sa].

where we regard $\mathcal{U}_n \cap \mathcal{U}_m = \emptyset$ $(n \neq m)$ as sets of vertices of $TN(\mathcal{U})$ even if $\mathcal{U}_n \cap \mathcal{U}_m \neq \emptyset$ as collections of open sets,[2] whence

$$N(\mathcal{U}_n \cup \mathcal{U}_{n+1}) \cap N(\mathcal{U}_{n+1} \cup \mathcal{U}_{n+2}) = N(\mathcal{U}_{n+1}).$$

Theorem 1. *A metric space $X = (X, d)$ is an ANR if and only if X has a zero-sequence $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$ of open covers with a map $f\colon |TN(\mathcal{U})| \to X$ satisfying the following conditions:*

(i) *$f(U) \in U$ for each $U \in TN(\mathcal{U})^{(0)} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$, and*
(ii) *$\lim_{n \to \infty} \operatorname{mesh}\{f(\sigma) \mid \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\} = 0$.*

Under the above circumstances, if the image $f(|TN(\mathcal{U})|)$ is always contained in $Y \subset X$, then Y is homotopy dense in X.

This characterization of ANR's is due to Nguyen To Nhu [N] (cf. [NS]). By the alternative proof given in [Sa], the additional assertion was obtained. As a corollary, we have the following:

Corollary 1. *Let X be an ANR (resp. an AR) contained in a metric space M. Then, there exists a G_δ-set $Z \subset M$ such that Z is an ANR (resp. an AR) and X is homotopy dense in Z.*

We can also apply Theorem 1 to find conditions such that the metric completion of a metric space X is an ANR with X a homotopy dense subset. A subset D of a metric space X is said to be *δ-dense* in X if $\operatorname{dist}(x, D) < \delta$ for every $x \in X$.

Corollary 2. *Let X be a metric space which has a zero-sequence $\mathcal{U} = (\mathcal{U}_n)_{n \in \mathbb{N}}$ of open covers with a map $f\colon |TN(\mathcal{U})| \to X$ satisfying the conditions (i) and (ii) of Theorem 1, where suppose $\mathcal{U}_n = \{B_X(x, \gamma_n) \mid x \in D_n\}$ for some δ_n-dense subset $D_n \subset X$ and $0 < \delta_n < \gamma_n$. Then, any metric space Z containing X isometrically as a dense subset is an ANR and X is homotopy dense in Z. In particular, the metric completion $\widetilde{X}$ of X is an ANR and X is homotopy dense in $\widetilde{X}$.*

In the above, note that the γ_n-dense subset D_n of X may not be δ_n-dense in Z. For example, $D_n = \{i/n \mid 1 \leqslant i < n\}$ is $1/n$-dense in $(0,1)$ but it is not $1/n$-dense in $[0,1]$.

Now, we consider the following extension property:

$(e)_k$ There exist constants $\alpha > 0$ and $\beta > 1$ such that every map $f\colon |K^{(k)}| \to X$ of the k-skeleton of an arbitrary simplicial complex K with $\operatorname{mesh}\{f(\sigma^{(k)}) \mid \sigma \in K\} < \alpha$ extends to a map $\tilde{f}\colon |K| \to X$ such that $\operatorname{diam} \tilde{f}(\sigma) \leqslant \beta \operatorname{diam} f(\sigma^{(k)})$ for each $\sigma \in K$.

[2] In [NS], we did not regard like this. Considering the set $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n \times \{n\}$ as the set of vertices of $NT(\mathcal{U})$, this is reasonable.

The following corollary is motivated by the proof of AR property of the hyperspaces (cf. [vM, §5.3]).

Corollary 3. *Every LC^{k-1} metric space X with the property $(e)_k$ is an ANR. In particular, a metric space X with $(e)_0$ is an ANR (cf. Theorem 3).*

Remark. In Theorem 1, X is an AR when $\mathcal{U}_1 = \{X\}$. Every C^{k-1} and LC^{k-1} metric space X is an AR if it has the following:

- $(\tilde{e})_k$ there exists a constant $\beta > 1$ such that every map $f\colon |K^{(k)}| \to X$ of the k-skeleton of an arbitrary simplicial complex K extends to a map $\tilde{f}\colon |K| \to X$ such that $\operatorname{diam} \tilde{f}(\sigma) \leqslant \beta \operatorname{diam} f(\sigma^{(k)})$ for each $\sigma \in K$.

2. Uniform ANR's

In [Mi$_2$], E. Michael introduced uniform AR's and uniform ANR's, and studied them. Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be metric spaces and $A \subset X$. A map $f\colon X \to Y$ is said to be *uniformly continuous* at A if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a \in A$, $x \in X$ and $d_X(a,x) < \delta$ then $d_Y(f(a), f(x)) < \varepsilon$. A neighborhood U of A in X is called a *uniform neighborhood* if $\bigcup_{a\in A} B_X(a,\delta) \subset U$ for some $\delta > 0$. A metric space Y is called a *uniform* ANR if, for an arbitrary metric space X and a closed set $A \subset X$, every uniformly continuous map $f\colon A \to Y$ extends to a map $\tilde{f}\colon U \to Y$ from some uniform neighborhood U of A in X which is uniformly continuous at A. When f always extends over X (i.e., $U = X$), Y is a *uniform* AR. By virtue of [Mi$_2$, Theorem 1.2], a metric space Y is a uniform ANR (resp. a uniform AR) if and only if, for an arbitrary metric space Z which contains Y isometrically as a closed subset, there exists a retraction $r\colon U \to Y$ for some uniform neighborhood U in Y in Z (resp. $r\colon Z \to Y$) which is uniformly continuous at Y.[3] The concept of uniform ANR's is useful since the metric completion of every uniform ANR is also a uniform ANR.

By using a zero-sequence of open covers in §1, we can prove the following version of Proposition 1.4 in [Mi$_2$]:

Theorem 2. *For an arbitrary metric space X, the following are equivalent:*

- (a) *X is a uniform ANR;*
- (b) *Every metric space Z containing X isometrically as a dense subset is a uniform ANR and X is homotopy dense in Z;*
- (c) *X is isometrically embedded in some uniform ANR Z as a homotopy dense subset.*

[3] Such a retraction is called a *regular retraction* by H. Toruńczyk in [To$_2$].

Theorem 2 above means that a metric space X is a uniform ANR if and only if the metric completion of X is a uniform ANR with X a homotopy dense subset. However, in order that the metric completion of a metric ANR X is an ANR with X a homotopy dense subset, it is not necessary that X is a uniform ANR. In case X is totally bounded, X is a uniform ANR if and only if the metric completion of X is an ANR with X a homotopy dense subset.

A metric space Y is said to be *uniformly* LC^k if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that any map $f\colon \mathbf{S}^i \to Y$ with $\operatorname{diam} f(\mathbf{S}^i) < \delta$ extends to a map $\tilde{f}\colon \mathbf{B}^{i+1} \to Y$ with $\operatorname{diam} \tilde{f}(\mathbf{B}^{i+1}) < \varepsilon$ for every $i \leqslant k$. In stead of "uniformly LC^0", we also say "uniformly locally path-connected". The subspace of $\mathbb{R}^2$ in the example above is not uniformly locally path-connected.

Theorem 3. *Every uniformly LC^{k-1} metric space Y with the property $(e)_k$ is a uniform ANR. In particular, a metric space X with $(e)_0$ is a uniform ANR.*

By Theorems 2 and 3, we have the following variation of Corollary 3 (cf. [SU, Lemma 2]):

Corollary 4. *Let X be a metric space and Y a dense subset of X. If Y is uniformly LC^{k-1} and has the property $(e)_k$, then X and Y are uniformly ANR's and Y is homotopy dense in X.*

Remark. In Theorem 3 and Corollary 4, by replacing the property $(e)_k$ with $(\tilde{e})_k$ and adding the condition that Y is C^{k-1}, "uniform ANR" can be "uniform AR". In particular, a metric space X with $(\tilde{e})_0$ is a uniform AR.

3. Dense (or uniform) local hyper-connectedness

By using the notion of (local) hyper-connectedness, C.R. Borges [Bo] and R. Cauty [Ca] characterized AR's and ANR's, respectively. Here is introduced a little weaker notion. By Δ^{n-1}, we denote the standard $(n-1)$-simplex in $\mathbb{R}^n$, that is,

$$\Delta^{n-1} = \{(t_1, \dots, t_n) \in \mathbb{R}^n \mid t_i \geqslant 0,\ \textstyle\sum_{i=1}^{n+1} t_i = 1\}.$$

For an open cover $\mathcal{U}$ of a space X and $Y \subset X$, we denote

$$Y^n(\mathcal{U}) = \{(y_1, \dots, y_n) \in Y^n \mid \exists U \in \mathcal{U} \text{ such that } \{y_1, \dots, y_n\} \subset U\}.$$

It is said that a space X is *densely locally hyper-connected* if X has an open cover $\mathcal{W}$, a dense subset D and functions $h_n\colon D^n(\mathcal{W}) \times \Delta^{n-1} \to X$, $n \in \mathbb{N}$, which satisfy the following conditions:

(i) if $t_i = 0$ then

$$\begin{aligned} &h_n(y_1, \dots, y_n; t_1, \dots, t_n) \\ &\qquad = h_{n-1}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n); \end{aligned}$$

(ii) $\Delta^{n-1} \ni (t_1, \dots, t_n) \mapsto h_n(y_1, \dots, y_n; t_1, \dots, t_n) \in X$ is continuous for each $(y_1, \dots, y_n) \in D^n(\mathcal{W})$;

(iii) every open cover $\mathcal{U}$ of X has an open refinement $\mathcal{V}$ such that $\mathcal{V} \prec \mathcal{W}$ (hence $D^n(\mathcal{V}) \subset D^n(\mathcal{W})$) and

$$\{h_n((D \cap V)^n \times \Delta^{n-1}) \mid V \in \mathcal{V}\} \prec \mathcal{U} \quad \text{for each } n \in \mathbb{N}.$$

It should be noticed that each h_n need not be continuous. If $\mathcal{W}$ can be taken as $\mathcal{W} = \{X\}$ (i.e., $D^n(\mathcal{W}) = D^n$), we say that X is *densely hyper-connected.* In case $D = X$ (resp. $D = X$ and $\mathcal{W} = \{X\}$), X is *locally hyper-connected*[4] (resp. *hyper-connected*). This concept is very similar to Michael's convex structure in [Mi$_1$]. In [Bo] and [Ca], AR's and ANR's are characterized by the hyper-connectedness and the local hyper-connectedness, respectively. A similar characterization was obtained by Himmelberg [Hi] (cf. Curtis [Cu]). These characterizations can be generalized in terms of the dense hyper-connectedness as follows:

Theorem 4. *A metrizable space X is an ANR if and only if X is densely locally hyper-connected. Moreover, X is an AR if and only if X is densely hyper-connected.*

Remark. In the definition of densely local hyper-connectedness, if the images of functions h_n are contained in Y, then Y is homotopy dense in X. In fact, if the images of functions h_n are contained in Y, then $f(|TN(\mathcal{U})|) \subset Y$, hence Y is homotopy dense in X by the additional statement of Theorem 1.

For a metric space X and $\eta > 0$, we denote

$$X^n(\eta) = \{(x_1, \dots, x_n) \in X^n \mid \operatorname{diam}\{x_1, \dots, x_n\} < \eta\}.$$

A metric space X is said to be *uniformly locally hyper-connected* if there are $\eta > 0$ and functions $h_n : X^n(\eta) \times \Delta^{n-1} \to X$, $n \in \mathbb{N}$, which satisfy the same conditions as (i) and (ii) above, and the following (iii') instead of (iii):

(iii') for each $\varepsilon > 0$, there is $0 < \delta < \varepsilon$ such that

$$\operatorname{diam} h_n(\{x\} \times \Delta^{n-1}) < \varepsilon \quad \text{for every } n \in \mathbb{N} \text{ and } x \in X^n(\delta).$$

When every h_n is defined on the whole space $X^n \times \Delta^{n-1}$, it is said that X is *uniformly hyper-connected.*

Now, we give a characterization of uniform ANR's and uniform AR's.

[4] The local hyper-connectedness is in the sense of [Ca] but not in the sense of [Bo].

Theorem 5. *A metric space $X = (X, d)$ is a uniform ANR if and only if X is uniformly locally hyper-connected. Moreover, X is a uniform AR if and only if X is uniformly hyper-connected.*

The following is a combination of Theorems 2 and 5:

Corollary 5. *Let X be a uniformly (locally) hyper-connected metric space and Z a metric space which contains X isometrically as a dense subset. Then, X and Z are uniform AR's (uniform ANR's) and X is homotopy dense in Z. In particular, the metric completion $\tilde{X}$ of X is a uniform AR (uniform ANR) and X is homotopy dense in $\tilde{X}$.*

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